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Mathematics

Hedging of European Option of Integral Type

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ABSTRACT. The problem of European Option hedging is considered. The integral type option is investigated in the case of Bachelier financial market model. We solve this hedging problem using the local time of the risky asset price process and its relationships with the payoffs of option. At first, we give the Clark stochastic integral representation formula in an explicit form for the local time and then we use the Trotter-Meyer Theorem and the Fubini Theorem of stochastic type. It is well known that the Clark-Ocone stochastic integral representation formula is the effective tool for solving of hedging problem. But in our case there are some difficulties to use this formula directly, because integrands of integral type payoffs are not differentiable by Malliavin. In the Malliavin theory it is well known that the indicator of event A is Malliavin differentiable if and only if probability $P(A)$ is equal to zero or one. Hence, for all t the indicator $I_{\{a \le w_t \le b\}}$ does not have Malliavin derivative. We prove that if the square integrable random process is not stochastic differentiable, then the "average" process is not stochastic differentiable either. For the check of the mentioned proposition we use one result proved by us: if square integrable random processes u_t has the Wiener-Chaos decomposition with kernels $f_{u,n}^t(\cdot)$, measurable in all their variables, then the average process with respect to dt has the Wiener-Chaos decomposition with kernels coinciding with the average of $f_{u,n}^t(\cdot)$ with respect to dt. Moreover, we need calculation of some integrals connected with the normal distribution and for completeness of a statement we give calculation of these integrals in Appendix. *© 2014 Bull. Georg. Natl. Acad. Sci.*

Key words: *Bachelier model, Clark-Ocone representation, local time, Trotter-Meyer theorem, Fubini theorem, hedging problem.*

We consider the European Option of integral type in the case of Bachelier market model. We develop the method of hedging of this option based on the application the local time of the risky asset price S. We give the Clark representation of local time and then using the relation between the payoffs of option and local times based on the stochastic type Fubini theorem we obtain the Clark integral representation of payoffs of our option. Therefore we solve the hedging problem. The method will be useful in the cases, when there are difficulties to use directly the Clark-Ocone integral representation [1, 2].

Let on the probability space $(\Omega, \Im, P)$ be given the Wiener process $w = (w_t)$, $t \in [0,T]$ and $(\Im_t^w)$, $t \in [0,T]$ be the natural filtration generated by the Wiener process w. Consider the Bachelier market model (see, for example [3]) with the risk-free asset price evolution described by

$$B_t \equiv 1, \tag{1}$$

and risky asset price evolution

$$dS_t = \mu dt + \sigma dw_t, \quad S_0 = 1, \tag{2}$$

where $\mu \in R$ is appreciation rate and $\sigma > 0$ is volatility coefficient.

Let

$$Z_T = \exp\{-\frac{\mu}{\sigma} w_T - \frac{1}{2}(\frac{\mu}{\sigma})^2 T\}$$

and $\overline{P}_T$ is the measure on $(\Omega, \Im_T^w)$ such that

$$d\overline{P}_T = Z_T dP .$$

From Girsanov's Theorem [3] it follows that under this measure (martingale risk neutral measure)

$$\overline{w}_t = w_t + \frac{\mu}{\sigma} t$$

there is the standard Wiener process and

$$dS_t = \sigma d\overline{w}_t, \quad S_0 = 1 \tag{3}$$

or

$$S_t = 1 + \sigma \overline{w}_t .$$

Consider the problem of "replication" the European Option with the payoff of integral type

$$G = \int_0^T I_{\{a \le S_t \le b\}} \sigma^2 dt \tag{4}$$

(where a and b are some positive constants, $a < b$), i.e., one needs to find a trading strategy (β_t, γ_t), $t \in [0,T]$ such that the capital process

$$X_t = \beta_t B_t + \gamma_t S_t, \quad X_T = G \tag{5}$$

under the self-financing condition

$$dX_t = \beta_t dB_t + \gamma_t dS_t . \tag{6}$$

From the relations (3), (5) and (6) we have

$$G = X_T = X_0 + \int_0^T \sigma \gamma_t d\overline{w}_t . \tag{7}$$

Our problem is to find the trading strategy $(\beta, \gamma) = (\beta_t, \gamma_t)$, $t \in [0,T]$. It is well-known that this problem is equivalent to finding a martingale representation of the payoff G with explicit form of integrand. Note that G is square integrable but not differentiable in Malliavin sense functional of Wiener process $\overline{w} = (\overline{w}_t)$, $t \in [0,T]$ and therefore we try to obtain the Clark integral representation with known integrand applying a nonconventional method (because the Clark-Ocone's well-known method here is not applicable).

Consider the local time of stochastic process S_t, $t \in [0,T]$. By the definition ([4], IV.44.1) local time of S at the point $x \in R$ is

$$l_t^x(S) = |S_t - x| - |S_0 - x| - \int_0^t \operatorname{sgn}(S_u - x)dS_u . \qquad (8)$$

For any measurable and bounded real function $\varphi(x)$ ([4], Trotter-Meyer Theorem IV.45.1) the following relation is true

$$\int_0^T \varphi(S_t)d\langle S\rangle_t = \int_{-\infty}^{\infty} l_T^x(S)\varphi(x)dx , \qquad (9)$$

where $\langle S\rangle_t$ is the predictable square variation of the martingale S_t, $t \in [0,T]$.

Suppose that

$$\varphi(x) = I_{\{a \le x \le b\}} .$$

Note also that, according to the Ito's formula

$$dS_t^2 = \sigma^2 dt + 2\sigma S_t d\overline{w}_t ,$$

and, hence,

$$\langle S\rangle_t = \sigma^2 t .$$

Further, from the relation (9) we obtain

$$\int_0^T I_{\{a \le S_t \le b\}}\sigma^2 dt = \int_a^b l_T^x(S)dx . \qquad (10)$$

Auxiliary results

In the Malliavin theory it is well known that the indicator of event A is Malliavin differentiable if and only if probability $P(A)$ is equal to zero or one (see Proposition 1.2.6 [5]). Hence, the indicator $I_{\{a \le w_t \le b\}}$ has no Malliavin derivative. We prove that if the square integrable random process is not stochastic differentiable, then the "average" process is not stochastic differentiable either. At first we will formulate one result proved by us:

Theorem 1. *If square integrable random processes* $u_t \in L_2([0,T]\times\Omega)$ *has the Wiener-Chaos decomposition* $u_t = \sum_{n=0}^{\infty} I_n(f_n^t(\cdot))$ *with kernels, measurable in all their variables, then the average process* $\int_0^T u_t dt$ *has the development*

$$\int_0^T u_t dt = \sum_{n=0}^{\infty} I_n\left(\int_0^T f_n^t(\cdot)dt\right) .$$

Theorem 2. *Let the square integrable random processes* $u_t \in L_2([0,T]\times\Omega)$ *such that for almost all* $t \in [0,T]$ *the random variable* u_t *does not belong to* $D_{2,1}$. *Then the average process* $\int_0^T u_t dt$ *is not in the* *space* $D_{2,1}$.

Proof. For almost all $t \in [0,T]$ u_t is square integrable random variable, and hence it has the Wiener-Chaos decomposition

$$u_t = \sum_{n=0}^{\infty} I_n(f_n^t(\cdot)),$$

where the deterministic kernels $f_n^t(\cdot)$ are symmetric and depend on the parameter *t*. Using the standard approximation technique to the process u_t in $L_2([0,T]\times\Omega)$ by a sequence of simple processes, these kernels can be chosen to be measurable in all their variables. Hence, due to Theorem 1 we have

$$\int_0^T u_t dt = \sum_{n=0}^{\infty} I_n(\int_0^T f_n^t(\cdot)dt).$$

Further, according to the Proposition 1.2.7 [5], the series

$$\sum_{n=1}^{\infty} nn! \| f_n^t(\cdot) \|_{L_2([0,T]^n)}$$

is unconvergent for all $t \in [0,T]$.

On the other hand, according to the Fubini theorem, we can conclude that the series

$$\sum_{n=1}^{\infty} \int_0^T nn! \| f_n^t(\cdot) \|_{L_2([0,T]^n)} dt = \int_0^T \left[\sum_{n=1}^{\infty} nn! \| f_n^t(\cdot) \|_{L_2([0,T]^n)} \right] dt$$

is also unconvergent, because otherwise we obtain that

$$\sum_{n=1}^{\infty} nn! \| f_n^t(\cdot) \|_{L_2([0,T]^n)} < \infty$$

for almost all $t \in [0,T]$.

Therefore, using again the Proposition 1.2.7 [5], we easily ascertain that the theorem is true.

Corollary. Since the indicator function $I_{\{a \le w_t \le b\}}$ does not belong to $D_{2,1}$ for all $t \in [0,T]$, hence, for all real number $a < b$

$$\int_0^T I_{\{a \le w_t \le b\}} dt \notin D_{2,1}.$$

Below we need calculation of some integrals connected with the normal distribution, whose value will be given below in the form of propositions. For the sake of complements, we included a proof of these propositions in the Appendix.

Proposition 1. *For any constants* $c_1 \in R$, $c_2 > 0$ *and* $c > 0$ *the following relationships are fulfilled:*

i)

$$\int_\alpha^\beta \Phi_{0,1}(\frac{x+c_1}{c_2})dx = [(x+c_1)\Phi_{0,1}(\frac{x+c_1}{c_2}) + c_2\varphi_{0,1}(\frac{x+c_1}{c_2})] |_\alpha^\beta ;$$

ii)

$$\int_\alpha^\beta x\Phi_{0,1}(\frac{x}{c})dx = \frac{1}{2}[(x^2 - c^2)\Phi_{0,1}(\frac{x}{c}) + cx\varphi_{0,1}(\frac{x}{c})] |_\alpha^\beta .$$

Proposition 2. *The following relationship is true*

$$E\,|\,w_T - x\,| = x[2\Phi_{0,1}(\frac{x}{\sqrt{T}}) - 1] + 2\sqrt{T}\varphi_{0,1}(\frac{x}{\sqrt{T}}).$$

Integral representation theorem and hedging of the option

Now we investigate the hedging problem of European Option with payoff

$$U = \int_0^T I_{\{a \le S_t \le b\}} dt, \tag{11}$$

where

$$S_t = 1 + \mu t + \sigma w_t$$

is risky asset price and risky free asset price $B_t \equiv 1$, i. e., we consider the Bachelier model. Note that U_T is really the occupation time of (a, b) up to time T of risky asset process S.

Under the martingale measure $\overline{P}$ ($\overline{P} \sim P$ and is such that [3] $d\overline{P} = Z_T dP$ with $Z_T = \exp\{-\frac{\mu}{\sigma} w_T - \frac{1}{2}(\frac{\mu}{\sigma})^2 T\}$)

$$\overline{w}_t = w_t + \frac{\mu}{\sigma} t$$

is the standard Wiener process and

$$S_t = 1 + \sigma \overline{w}_t. \tag{12}$$

From (11), using (12), we have

$$U = \int_0^T I_{\{\frac{a-1}{\sigma} \le \overline{w}_t \le \frac{b-1}{\sigma}\}} dt. \tag{13}$$

According to the Trotter-Meyer Theorem ([4], Theorem IV.45.1) for any measurable and bounded real function $\varphi(x)$ the following relation holds

$$\int_0^T \varphi(S_t) d\langle S \rangle_t = \int_{-\infty}^{\infty} l_T^x(S) \varphi(x) dx,$$

where $\langle S \rangle_t$ is the predictable square variation of martingale S and $l_T^x(S)$ is the local time of S at the point $x \in R$. If we take here $\varphi(x) = I_{\{a \le x \le b\}}$, then for the Wiener process we obtain that

$$U = \int_0^T I_{\{\frac{a-1}{\sigma} \le \overline{w}_t \le \frac{b-1}{\sigma}\}} dt = \int_{(a-1)/\sigma}^{(b-1)/\sigma} l_T^x(\overline{w}) dx, \tag{14}$$

where $l_T^x(\overline{w})$ is the local time of the Wiener process $\overline{w}$ at the point $x \in R$.

It is well known ([6] or [4], Tanaka's formula IV.43) that the local time of the Wiener process admits the following representation:

$$l_T^x(\overline{w}) = |\,\overline{w}_T - x\,| - |\,x\,| - \int_0^T \operatorname{sgn}(\overline{w}_t - x) d\overline{w}_t. \tag{15}$$

Theorem 3. *The local time of Wiener process admits the following integral representation*

$$l_T^x(\overline{w}) = \overline{E} \mid \overline{w}_T - x \mid - \mid x \mid + \int_0^T \{1 - 2\Phi_{0,1}(\frac{x - \overline{w}_t}{\sqrt{T-t}})\} d\overline{w}_t - \int_0^T \mathrm{sgn}(\overline{w}_t - x) d\overline{w}_t \,, \tag{16}$$

where

$$\overline{E} \mid \overline{w}_T - x \mid = x[2\Phi_{0,1}(\frac{x}{\sqrt{T}}) - 1] + 2\sqrt{T}\varphi_{0,1}(\frac{x}{\sqrt{T}}) \,. \tag{17}$$

Proof. Due to the Clark-Ocone integral representation formula [2], we have

$$\mid \overline{w}_T - x \mid = \overline{E} \mid \overline{w}_T - x \mid + \int_0^T \overline{E}[D_t(\mid \overline{w}_T - x \mid) \mid \Im_t^{\overline{w}}] d\overline{w}_t \,.$$

Using the relation ([5], Proposition 1.2.4 or [7], Theorem 2)

$$D_t(\mid \overline{w}_T - x \mid) = D_t[(\overline{w}_T - x)^+ + (\overline{w}_T - x)^-] = I_{\{\overline{w}_T \ge x\}} - I_{\{\overline{w}_T < x\}}$$

the integrand of the Clark-Ocone integral representation we can rewrite as

$$\overline{E}[D_t(\mid \overline{w}_T - x \mid) \mid \Im_t^{\overline{w}}] = \overline{E}[I_{\{\overline{w}_T \ge x\}} \mid \Im_t^{\overline{w}}] - \overline{E}[I_{\{\overline{w}_T < x\}} \mid \Im_t^{\overline{w}}] := J_1 - J_2 \,.$$

According to the well-known properties of Wiener process and conditional mathematical expectation it is not difficult to see that

$$J_1 := \overline{E}[I_{\{\overline{w}_T \ge x\}} \mid \Im_t^{\overline{w}}] = \overline{E}[I_{\{\overline{w}_T - \overline{w}_t + \overline{w}_t \ge x\}} \mid \overline{w}_t] = \overline{E}[I_{\{\overline{w}_T - \overline{w}_t + y \ge x\}}] \mid_{y = \overline{w}_t} =$$

$$= P\{\overline{w}_T - \overline{w}_t \ge x - y\} \mid_{y = \overline{w}_t} = 1 - \Phi_{0,T-t}(x - y) \mid_{y = \overline{w}_t} = 1 - \Phi_{0,1}\left(\frac{x - \overline{w}_t}{\sqrt{T-t}}\right).$$

Analogously, we can verify that

$$J_2 := \overline{E}[I_{\{\overline{w}_T < x\}} \mid \Im_t^{\overline{w}}] = \Phi_{0,1}\left(\frac{x - \overline{w}_t}{\sqrt{T-t}}\right).$$

Combining now the above obtained expressions and using the relation (15) and Proposition 2 we easily complete the proof of theorem.

Theorem 4. T*he following integral representation formula is fulfilled*

$$U = \overline{E}U + \int_0^T L_t d\overline{w}_t \,,$$

where $\overline{E}(\cdot)$ *is mathematical expectation with respect to* $\overline{P}$,

$$\overline{E}U = \{(x^2 + T)\Phi_{0,1}(\frac{x}{\sqrt{T}}) + Tx\varphi_{0,1}(\frac{x}{\sqrt{T}}) - \frac{1}{2}x^2[\mathrm{sgn}(x) + 1]\} \mid_{(a-1)/\sigma}^{(b-1)/\sigma} \tag{18}$$

and

$$L_t = \{x[1 - \mathrm{sgn}(\overline{w}_t - x)] - 2(x - \overline{w}_t)\Phi_{0,1}(\frac{x - \overline{w}_t}{\sqrt{T-t}}) - 2\sqrt{T-t}\varphi_{0,1}(\frac{x - \overline{w}_t}{\sqrt{T-t}})\} \mid_{(a-1)/\sigma}^{(b-1)/\sigma} .$$

Proof. From (14) using the relations (16) and (17), we obtain that

$$U = \int_{(a-1)/\sigma}^{(b-1)/\sigma} \{x[2\Phi_{0,1}(\frac{x}{\sqrt{T}}) - 1] + 2\sqrt{T}\varphi_{0,1}(\frac{x}{\sqrt{T}}) - \mid x \mid\} dx +$$

$$+\int_{(a-1)/\sigma}^{(b-1)/\sigma}\{\int_0^T[1-\Phi_{0,1}(\frac{x-\overline{w}_t}{\sqrt{T-t}})-\operatorname{sgn}(\overline{w}_t-x)]d\overline{w}_t\}dx.$$

Fubini Theorem of stochastic type ([6], Lemma III.4.1 or [8], Corollary of the Lemma IV.2.4) give us possibility to have the following representation:

$$U=\int_{(a-1)/\sigma}^{(b-1)/\sigma}[2x\Phi_{0,1}(\frac{x}{\sqrt{T}})+2\sqrt{T}\varphi_{0,1}(\frac{x}{\sqrt{T}})-(x+|x|)]dx+$$

$$+\int_0^T\{\int_{(a-1)/\sigma}^{(b-1)/\sigma}[1-\Phi_{0,1}(\frac{x-\overline{w}_t}{\sqrt{T-t}})-\operatorname{sgn}(\overline{w}_t-x)]dx\}d\overline{w}_t.$$

Using the Proposition 1 ii) with $\alpha=(a-1)/\sigma$, $\beta=(b-1)/\sigma$ and $c=\sqrt{T}$ we have

$$\int_{(a-1)/\sigma}^{(b-1)/\sigma}2x\Phi_{0,1}(\frac{x}{\sqrt{T}})dx=(x^2-T)\Phi_{0,1}(\frac{x}{\sqrt{T}})\Big|_{(a-1)/\sigma}^{(b-1)/\sigma}+Tx\varphi_{0,1}(\frac{x}{\sqrt{T}})\Big|_{(a-1)/\sigma}^{(b-1)/\sigma}.$$

Further, it is easy to see that

$$\int_{(a-1)/\sigma}^{(b-1)/\sigma}2\sqrt{T}\varphi_{0,1}(\frac{x}{\sqrt{T}})dx=2\sqrt{T}\cdot\sqrt{T}\int_{(a-1)/\sigma}^{(b-1)/\sigma}\frac{d}{dx}[\Phi_{0,1}(\frac{x}{\sqrt{T}})]dx=2T\Phi_{0,1}(\frac{x}{\sqrt{T}})\Big|_{(a-1)/\sigma}^{(b-1)/\sigma}.$$

Moreover, we have

$$\int_{(a-1)/\sigma}^{(b-1)/\sigma}(x+|x|)dx=\frac{1}{2}x^2[\operatorname{sgn}(x)+1]\Big|_{(a-1)/\sigma}^{(b-1)/\sigma}.$$

On the other hand, due to the Proposition 1 i) with $\alpha=(a-1)/\sigma$, $\beta=(b-1)/\sigma$, $c_1=-\overline{w}_t$, $c_2=\sqrt{T-t}$, using the relation

$$\int_{(a-1)/\sigma}^{(b-1)/\sigma}\operatorname{sgn}(c-x)dx=x\operatorname{sgn}(c-x)\Big|_{(a-1)/\sigma}^{(b-1)/\sigma},$$

it is not difficult to see that

$$\int_{(a-1)/\sigma}^{(b-1)/\sigma}[1-\Phi_{0,1}(\frac{x-\overline{w}_t}{\sqrt{T-t}})-\operatorname{sgn}(\overline{w}_t-x)]dx=[x-x\operatorname{sgn}(\overline{w}_t-x)]\Big|_{(a-1)/\sigma}^{(b-1)/\sigma}-$$

$$-2(x-\overline{w}_t)\Phi_{0,1}(\frac{x-\overline{w}_t}{\sqrt{T-t}})\Big|_{(a-1)/\sigma}^{(b-1)/\sigma}-2\sqrt{T-t}\varphi_{0,1}(\frac{x-\overline{w}_t}{\sqrt{T-t}})\Big|_{(a-1)/\sigma}^{(b-1)/\sigma}.$$

Combining now the above obtained expressions we complete the proof of theorem.

In the conclusion we notice that the components of hedging strategy are: $\gamma_t=L_t/\sigma$ and $\beta_t=X_t-\gamma_t S_t$, where the capital process

$$X_t=\overline{E}U+\int_0^t L_s d\overline{w}_s.$$

and the price of this option is defined by (18).

Remark. *Here we do not consider the problem of choosing constants a and b. We note only that if b < 1, then* $(x+|x|)I_{[(a-1)/\sigma,(b-1)/\sigma]}(x)=0$ *and therefore the price of our option will be equal to*

$$\overline{E}U=\{(x^2+T)\Phi_{0,1}(\frac{x}{\sqrt{T}})+Tx\varphi_{0,1}(\frac{x}{\sqrt{T}})\}\Big|_{(a-1)/\sigma}^{(b-1)/\sigma}.$$

Appendix

Proof of Proposition 1. i) Using the integration by parts formula, due to the well-known properties of the integration, it is not difficult to see that

$$\int_\alpha^\beta \Phi_{0,1}(\frac{x+c_1}{c_2})dx=[x\Phi_{0,1}(\frac{x+c_1}{c_2})]\Big|_\alpha^\beta-\frac{1}{c_2}\int_\alpha^\beta x\varphi_{0,1}(\frac{x+c_1}{c_2})dx=$$

$$=[x\Phi_{0,1}(\frac{x+c_1}{c_2})]\Big|_\alpha^\beta-\frac{1}{c_2}\int_\alpha^\beta(x+c_1-c_1)\varphi_{0,1}(\frac{x+c_1}{c_2})dx=$$

$$=[x\Phi_{0,1}(\frac{x+c_1}{c_2})]\Big|_\alpha^\beta+c_2\int_\alpha^\beta d[\varphi_{0,1}(\frac{x+c_1}{c_2})]+c_1\int_\alpha^\beta\varphi_{0,1}(\frac{x+c_1}{c_2})d(\frac{x+c_1}{c_2})=$$

$$=[(x+c_1)\Phi_{0,1}(\frac{x+c_1}{c_2})+c_2\varphi_{0,1}(\frac{x+c_1}{c_2})]\Big|_\alpha^\beta.$$

ii) Analogously, consecutively using the integration by parts formula, we easily ascertain that

$$\int_\alpha^\beta x\Phi_{0,1}(\frac{x}{c})dx=\frac{1}{2}\int_\alpha^\beta\Phi_{0,1}(\frac{x}{c})d(x^2)=\frac{1}{2}[x^2\Phi_{0,1}(\frac{x}{c})]\Big|_\alpha^\beta-$$

$$-\frac{1}{2c}\int_\alpha^\beta x^2\varphi_{0,1}(\frac{x}{c})dx=\frac{1}{2}[x^2\Phi_{0,1}(\frac{x}{c})]\Big|_\alpha^\beta+\frac{1}{2}c\int_\alpha^\beta xd[\varphi_{0,1}(\frac{x}{c})]=$$

$$=\frac{1}{2}[x^2\Phi_{0,1}(\frac{x}{c})]\Big|_\alpha^\beta+\frac{1}{2}c[x\varphi_{0,1}(\frac{x}{c})]\Big|_\alpha^\beta-\frac{1}{2}c^2\int_\alpha^\beta\varphi_{0,1}(\frac{x}{c})d(\frac{x}{c})=$$

$$=\frac{1}{2}[(x^2-c^2)\Phi_{0,1}(\frac{x}{c})+cx\varphi_{0,1}(\frac{x}{c})]\Big|_\alpha^\beta.$$

Proof of Proposition 2. By the definition of mathematical expectation, due to the well-known properties of normal distribution and integration, we can write

$$E|w_T-x|=\frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{+\infty}|u-x|\exp\{-\frac{u^2}{2T}\}du=$$

$$=\frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{x}(x-u)\exp\{-\frac{u^2}{2T}\}du+\frac{1}{\sqrt{2\pi T}}\int_{x}^{+\infty}(u-x)\exp\{-\frac{u^2}{2T}\}du=$$

$$=x\Phi_{0,T}(x)+\frac{T}{\sqrt{2\pi T}}\int_{-\infty}^{x}\exp\{-\frac{u^2}{2T}\}d(-\frac{u^2}{2T})-$$

$$-\frac{T}{\sqrt{2\pi T}}\int_{x}^{+\infty}\exp\{-\frac{u^2}{2T}\}d(-\frac{u^2}{2T})-x[1-\Phi_{0,T}(x)]=$$

$$= x\Phi_{0,T}(x) + T\varphi_{0,T}(u)\,|_{-\infty}^{x} - T\varphi_{0,T}(u)\,|_{x}^{+\infty} - x[1-\Phi_{0,T}(x)] =$$

$$= x[2\Phi_{0,1}(\frac{x}{\sqrt{T}}) - 1] + 2\sqrt{T}\varphi_{0,1}(\frac{x}{\sqrt{T}})\,.$$

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მათემატიკა

ინტეგრალური ტიპის ევროპული ოფციონის ჰეჯირება

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განიხილება ევროპული ოფციონის ჰეჯირების ამოცანა. გამოკვლეულია ინტეგრალური ტიპის ოფციონი ფინანსური ბაზრის ბაშელიეს მოდელის შემთხვევაში. ჰეჯირების აღნიშნული ამოცანა გადაწყვეტილია რისკიანი ფასიანი აქტივის პროცესის ლოკალური დროის ცნებისა და ოფციონის გადასახადის ფუნქციასთან მისი დამოკიდებულების გამოყენებით. პირველ რიგში ჩვენ გამოგვყავს კლარკის სტოქასტური ინტეგრალური წარმოდგენის ფორმულა ცხადი ინტეგრანდით და შემდგომ ვიყენებთ ტროტერ-მეიერის თეორემასა და სტოქასტური ტიპის ფუბინის თეორემას. კარგად არის ცნობილი, რომ ჰეჯირების ამოცანის გადაწყვეტის ეფექტურ საშუალებას წარმოადგენს კლარკ-ოკონის სტოქასტური ინტეგრალური წარმოდგენის ფორმულა. მაგრამ ჩვენ შემთხვევაში წარმოიქმნება გარკვეული სიძნელეები, ვინაიდან ინტეგრალური გადასახადის ფუნქციის ინტეგრანდი არაა დიფერენცირებადი მალივენის აზრით და კლარკ-ოკონის ფორმულის პირდაპირი გამოყენება შეუძლებელია. მალივენის აღრიცხვაში ცნობილია, რომ A ხდომილების ინდიკატორი არის მალივენის აზრით წარმოებადი მაშინ და მხოლოდ მაშინ, როდესაც ამ ხდომილების $P(A)$ ალბათობა ნული ან ერთია. შესაბამისად, ნებისმიერი t-სთვის $I_{\{a \le w_t \le b\}}$ ინდიკატორს არ გააჩნია მალივენის წარმოებული. ჩვენ ვაჩვენეთ, რომ თუ კვადრატით ინტეგრებადი სტოქასტური პროცესი არ არის სტოქასტურად წარმოებადი, მაშინ "გასაშუალებული" პროცესი აგრეთვე არ არის სტოქასტურად წარმოებადი. აღნიშნული დებულების შესამოწმებლად გამოყენებულ იქნა ჩვენ მიერ მიღებული ერთი შედეგი: თუ კვადრატით ინტეგრებად სტოქასტურ u_t პროცესს გააჩნია ვინერის ქაოტური

წარმოდგენა $f_{u,n}^{t}(\cdot)$ გულებით, რომელიც ზომადია ყველა თავისი ცვლადის მიმართ, მაშინ dt -ს მიმართ გასაშუალოებულ პროცესს აგრეთვე გააჩნია ვინერის ქაოტური წარმოდგენა გულებით, რომელიც ემთხვევა $f_{u,n}^{t}(\cdot)$ გულის საშუალოს dt-ს მიმართ. გარდა ამისა, ჩვენ დაგვჭირდა ნორმალურ განაწილებასთან დაკავშირებული ზოგიერთი ინტეგრალის გამოთვლა და ნაშრომის სისრულის მიზნით აღნიშნული ინტეგრალების გამოთვლა მოყვანილია დამატებაში.

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