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Physics

Calculation of the Gravitoelectromagnetism Force for the O'Hanlon-Tupper Spacetime

Morteza Yavari

Islamic Azad University, Kashan Branch, Kashan, Iran

(Presented by Academy Member Anzor Khelashvili)

ABSTRACT. By using the threading splitting concepts for a time dependent spacetime, the time dependent quasi-Maxwell equations in terms of the gravitoelectromagnetism fields are discussed. The motion of a test particle in the O'Hanlon-Tupper spacetime by applying the Hamilton-Jacobi method and the quasi-Maxwell equations is studied. Also, the gravitoelectromagnetism force in this spacetime is calculated. *© 2013 Bull. Georg. Natl. Acad. Sci.*

Key words: *O'Hanlon-Tupper spacetime, quasi-Maxwell equations, particle trajectory, gravitoelectromagnetism force.*

1. Introduction

The slicing and threading points of view are introduced, respectively, by Misner, Thorne and Wheeler [1] in 1973 and, Landau and Lifshitz [2] in 1975. Both points of view can be traced back when Landau and Lifshitz [3] in 1941 introduced the threading point of view splitting of the spacetime metric and in the stationary case connection to yield the spatial gravitational force. After them, Lichnerowicz [4] introduced the beginnings of the slicing point of view. Also, Møller [5] discussed the spatial gravitational force for a general spacetime. In 1956, Zel'manov [6] discussed the splitting of Einstein field equations in general case. For more details about these formalisms, see [7].

In threading point of view, splitting of spacetime is introduced by a family of timelike congruences with unit tangent vector field may be interpreted as the world-lines of a family of observers, and it defines a local time direction plus a local space through its orthogonal subspace in tangent space. Let $(\mathrm{M}, g_{\alpha\beta})$ be a 4-dim manifold of a stationary spacetime. The Greek indices run from 0 to 3 while the Latin indices take the values 1 to 3. We can construct a 3-dim orbit manifold as $\bar{\mathrm{M}} = \frac{\mathrm{M}}{\mathrm{G}}$ with projected metric tensor γ_{ij} by the smooth map $\Sigma : \mathrm{M} \to \bar{\mathrm{M}}$, where $\Sigma(p)$ denotes the orbit of the timelike Killing vector $\frac{\partial}{\partial t}$ at the point $p \in \mathrm{M}$ and G is 1-dim group of transformations generated by timelike Killing vector of the spacetime under consideration [7,8]. The threading decomposition leads to the following line element [2,8]:

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = h\left(dt - g_i dx^i\right)^2 - g_{ij}dx^i dx^j, \quad (1)$$

where $\gamma_{ij} = -g_{ij} + h g_i g_j$ in which $g_i = -\frac{g_{0i}}{h}$ and $h = g_{00}$. If we apply the time dependent γ_{ij} as the metric, then the vacuum Einstein field equations may be written as the time dependent quasi-Maxwell equations (c = G = 1) [6,9]:

$$^*\nabla \cdot {}^*\mathbf{E}_g = {}^*E_g^2 + \frac{1}{2}{}^*H_g^2 - \frac{^*\partial D}{\partial t} - d, \quad (2)$$

$$^*\nabla \times {}^*\mathbf{H}_g = 2\left({}^*\mathbf{S}_g + {}^*\mathbf{M}\right), \quad (3)$$

$$^*K_{ij} = -{}^*\nabla_{(i}{}^*E_{gj)} + {}^*E_{gi}{}^*E_{gj} + \frac{1}{2}\left({}^*H_{gi}{}^*H_{gj} - \gamma_{ij}{}^*H_g^2\right)$$
$$+ 2D_{ik}D_j^k + \sqrt{\gamma}\varepsilon_{nk(i}D_{j)}^n{}^*H_g^k - \frac{^*\partial D_{ij}}{\partial t} - DD_{ij}, \quad (4)$$

where $\frac{^*\partial}{\partial t} = \frac{1}{\sqrt{h}}\frac{\partial}{\partial t}$, $\gamma = \det(\gamma_{ij})$ and $d = D_{ij}D^{ij}$ such that

$$D_{ij} = \frac{1}{2}\frac{^*\partial\gamma_{ij}}{\partial t},\ D^{ij} = -\frac{1}{2}\frac{^*\partial\gamma^{ij}}{\partial t},\ D = \gamma^{ij}D_{ij} = \frac{^*\partial\ln\sqrt{\gamma}}{\partial t}. \quad (5)$$

The symbols () and [] represent the commutation and anticommutation over indices and the 3-dim Levi-Civita tensor ε_{ijk} is antisymmetric under interchange of any pair of indices such that $\varepsilon_{123} = \varepsilon^{123} = 1$ [2]. Also, divergence and curl of an arbitrary vector in a 3-space with metric γ_{ij} are defined as $^*\nabla \cdot \mathbf{A} = \frac{1}{\sqrt{\gamma}}\left(\sqrt{\gamma}A^i\right)_{*i}$ and $\left({}^*\nabla\times\mathbf{A}\right)^i = \frac{\varepsilon^{ijk}}{2\sqrt{\gamma}}A_{[k*j]}$ such that $_{*i} = {}^*\partial_i = \partial_i + g_i\frac{\partial}{\partial t}$. In equation (4), the 3-dim starry Ricci tensor $^*K_{ij}$ is constructed from 3-dim starry Christoffel symbols as $^*K_{ij} = {}^*\lambda_{ij*k}^k - {}^*\lambda_{ik*j}^k + {}^*\lambda_{ij}^n{}^*\lambda_{kn}^k - {}^*\lambda_{ik}^n{}^*\lambda_{nj}^k$ where $^*\lambda_{jk}^i = \frac{1}{2}\gamma^{il}\left(\gamma_{jl*k} + \gamma_{kl*j} - \gamma_{jk*l}\right)$ and the starry covariant derivatives of an arbitrary 3-vector are given by $^*\nabla_j A_i = A_{i*j} - {}^*\lambda_{ij}^k A_k$ and $^*\nabla_j A^i = A_{*j}^i + {}^*\lambda_{jk}^i A^k$. The time dependent gravitoelectromagnetism fields are defined in terms of the gravoelectric potential $\varphi = \ln\sqrt{h}$ and the gravomagnetic vector potential $\mathbf{g} = (g_1, g_2, g_3)$ as follows

$$^*\mathbf{E}_g = -{}^*\nabla\varphi - \frac{\partial\mathbf{g}}{\partial t}\ ;\ {}^*E_{gi} = -\varphi_{*i} - \frac{\partial g_i}{\partial t}, \quad (6)$$

$$\frac{^*\mathbf{H}_g}{\sqrt{h}} = {}^*\nabla\times\mathbf{g}\ ;\ \frac{^*H_g^i}{\sqrt{h}} = \frac{\varepsilon^{ijk}}{2\sqrt{\gamma}}g_{[k*j]}. \quad (7)$$

The gravitoelectromagnetism refers to a set of analogies between Maxwell equations and a reformulation of Einstein field equations in general relativity [10,11]. The vectors $^*\mathbf{S}_g = {}^*\mathbf{E}_g \times {}^*\mathbf{H}_g$ and $^*\mathbf{M}$ have components as $^*S_g^i = \frac{\varepsilon^{ijk}}{\sqrt{\gamma}}{}^*E_{gj}{}^*H_{gk}$ and $^*M^i = -{}^*\nabla_j D^{ij} + {}^*\partial^i D$ while $^*\nabla_k D^{ij} = D_{*k}^{ij} + {}^*\lambda_{nk}^i D^{jn} + {}^*\lambda_{nk}^j D^{in}$ and $^*\partial^i = \gamma^{ij}{}^*\partial_j$. For more details about applications of gravitoelectromagnetism fields, see [12-14].

2. Motion of a test particle in the O'Hanlon-Tupper spacetime

2.1. Calculation of the trajectory

We consider the O'Hanlon-Tupper metric in Cartesian coordinates as follows [15]:

$$ds^2 = dt^2 - t^n\left(dx^2 + dy^2 + dz^2\right), \quad (8)$$

where n is an unknown constant. At first, it is easy to see that all components of gravitoelectromagnetism fields and starry Christoffel symbols vanish. Therefore, the time-dependent quasi-Maxwell equations reduce to

$$d = -\frac{\partial \mathrm{D}}{\partial t}, \tag{9}$$

$$\mathrm{DD}_{ij} - 2\mathrm{D}_{ik}\mathrm{D}_j^k + \frac{\partial \mathrm{D}_{ij}}{\partial t} = 0. \tag{10}$$

The solutions of these equations for the metric (8) are

$$n = \frac{2}{3} \ \& \ 2. \tag{11}$$

As is known [16], the solution $n = 2$ corresponds to the Chitre-Hartle spacetime. The Chitre-Hartle metric was first introduced as a model background for the study of scalar particle creation in homogeneous and isotropic spacetimes [17]. The physical significance of this spacetime was first pointed out by Fischetti et al. [18], who showed it to be a cosmological solution of the Einstein equations modified to include one-loop quantum gravitational corrections arising from the quantum trace anomaly.

In continuation, we determine the trajectory of a test particle of mass m that is moving in the O'Hanlon-Tupper spacetime by using the Hamilton-Jacobi equation [19-21]. Therefore, this equation is of the form

$$t^n\left(\frac{\partial \mathrm{S}}{\partial t}\right)^2 - \left(\frac{\partial \mathrm{S}}{\partial x}\right)^2 - \left(\frac{\partial \mathrm{S}}{\partial y}\right)^2 - \left(\frac{\partial \mathrm{S}}{\partial z}\right)^2 - m^2 t^n = 0. \tag{12}$$

To solve this partial differential equation, we use the method of separation of variables for the Hamilton-Jacobi function as below

$$\mathrm{S}(x,y,z,t) = \mathrm{L}_x x + \mathrm{L}_y y + \mathrm{L}_z z + \mathrm{T}(t), \tag{13}$$

where L_x, L_y and L_z are arbitrary constants and can be identified respectively as the angular momentum components of test particle along x, y and z-directions. Substituting the ansatz (13) into the Hamilton-Jacobi equation, gives the following expression for the function T as

$$\mathrm{T} = \sigma \begin{cases} \dfrac{\sqrt{(m^2\sqrt[3]{t^2} + \mathrm{L}^2)^3}}{m^2}, & n = \dfrac{2}{3}, \\ \sqrt{m^2t^2 + \mathrm{L}^2} + \mathrm{L}\ln t - \\ \quad -\mathrm{L}\ln\left(2\mathrm{L}\sqrt{m^2t^2+\mathrm{L}^2} + 2\mathrm{L}^2\right), & n = 2 \end{cases} \tag{14}$$

in which $\mathrm{L}^2 = \mathrm{L}_x^2 + \mathrm{L}_y^2 + \mathrm{L}_z^2$ and $\sigma = \pm 1$. Let us now obtain the trajectory of test particle by considering the following relations [19-21]:

$$\frac{\partial \mathrm{S}}{\partial \mathrm{L}_x} = \text{constant}, \quad \frac{\partial \mathrm{S}}{\partial \mathrm{L}_y} = \text{constant},$$
$$\frac{\partial \mathrm{S}}{\partial \mathrm{L}_z} = \text{constant}. \tag{15}$$

Finally, after calculating and simplifying, the set of equations (15) changes to the following relations

$$\frac{x}{\mathrm{L}_x} = \frac{y}{\mathrm{L}_y} = \frac{z}{\mathrm{L}_z} =$$
$$= \frac{\sigma}{\mathrm{L}} \begin{cases} -\dfrac{3\mathrm{L}\sqrt{m^2\sqrt[3]{t^2}+\mathrm{L}^2}}{m^2}, & n = \dfrac{2}{3}, \\ 1 - \ln t + \\ + \ \ln\left(2\mathrm{L}\sqrt{m^2t^2+\mathrm{L}^2} + 2\mathrm{L}^2\right), & n = 2 \end{cases} \tag{16}$$

we have taken the constants in equations (15) to be zero without any loss of generality. Therefore, the trajectory of particle have been calculated.

2.2. Calculation of the gravitoelectromagnetism force

In a spacetime with time dependent metric (1), the gravitoelectromagnetism force acting on a test particle whose mass m due to time dependent gravitoelectromagnetism fields as measured by threading observers is described by the following

$${}^*\mathbf{F}_g = \frac{{}^*d\,{}^*\mathbf{p}}{dt} - \frac{m}{\sqrt{1 - {}^*v^2}}\left\{{}^*\mathbf{E}_g + {}^*\boldsymbol{v} \times {}^*\mathbf{H}_g + \boldsymbol{f}\right\}, \tag{17}$$

where ${}^*\mathrm{p}^i = \dfrac{m\,{}^*v^i}{\sqrt{1-{}^*v^2}}$ such that ${}^*v^2 = \gamma_{ij}\,{}^*v^i\,{}^*v^j$ in which ${}^*v^i = \dfrac{v^i}{\sqrt{h}\left(1-\mathrm{g}_k v^k\right)}$ while $v^i = \dfrac{dx^i}{dt}$. Also, the starry total derivative with respect to time is defined as follows

$$\frac{{}^*d}{dt} = \frac{{}^*\partial}{\partial t} + {}^*v^i\,{}^*\partial_i. \tag{18}$$

In the equation (17), the last term is defined as

$$f^i = -{}^*\lambda^i_{jk}\,{}^*v^j\,{}^*v^k - 2\mathrm{D}^i_k\,{}^*v^k. \tag{19}$$

As before, with employing the equations (16), we can obtain

$${}^*\boldsymbol{v} = -\frac{\sigma t^{-\frac{n}{2}}}{\sqrt{m^2 t^n + \mathrm{L}^2}}\mathbf{L}, \tag{20}$$

in which $\mathbf{L} = \mathrm{L}_x\hat{i} + \mathrm{L}_y\hat{j} + \mathrm{L}_z\hat{k}$. In the next step, by using the previous result, we lead to

$$\frac{m}{\sqrt{1-{}^*v^2}} = t^{-\frac{n}{2}}\sqrt{m^2 t^n + \mathrm{L}^2}. \tag{21}$$

The above radicand has no real extremals. Hence, there are no bound states and the particle cannot be trapped by the extended object with the O'Hanlon-Tupper geometry. To continue our analysis, we need to calculate the last term of the equation (17). Thus, we have

$$\boldsymbol{f} = \frac{n\sigma t^{-\frac{n+2}{2}}}{\sqrt{m^2 t^n + \mathrm{L}^2}}\mathbf{L}. \tag{22}$$

With the help of equations (20-22), after some calculations, we finally obtain

$${}^*\mathbf{F}_g = 0. \tag{23}$$

3. Conclusions

The classical motion of a test particle in the O'Hanlon-Tupper spacetime has been studied. We proved that the test particle can not be trapped by this gravitational field. Also, it was shown that the gravito-electromagnetism force acting on test particle in this spacetime vanishes.

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გრავიტოელექტრომაგნიტური ძალის გამოთვლა ო'ჰანლონ-ტუპერის სივრცე-დროში

მ. იავარი

ისლამური უნივერსიტეტი აზადი, ქაშანის განყოფილება, ქაშანი, ირანი

(წარმოდგენილია აკადემიის წევრის ა.ხელაშვილის მიერ)

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