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Mathematics

Approximation by Trigonometric Polynomials in Subspace of Weighted Grand Lebesgue Spaces

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ABSTRACT. The paper presents the direct and inverse theorems of trigonometric approximation in closure of $L_w^p\ (1<p<\infty)$ by the norm of weighted grand Lebesgue spaces. The rate of deviation by summation means of Fourier trigonometric series is derived.

Key words: *weighted grand Lebesgue space, direct and inverse theorems, approximation by trigonometric polynomials, linear summation methods.*

The paper deals with basic approximation problems such as direct and inverse theorems of constructive theory of functions in certain subspace of weighted grand Lebesgue space. The estimates of deviations from the operator defined by summing means of Fourier trigonometric series are derived.

Let $\mathbb{T}=(-\pi,\pi)$ and $1<p<\infty$, $\theta>0$. The weighted grand Lebesgue space of $L_w^{p),\theta}(\mathbb{T})$ is defined as a set of measurable functions for which the norm

$$\|f\|_{p),\theta,w}=\sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon^\theta}{2\pi}\int_{\mathbb{T}}|f(x)|^{p-\varepsilon}w(x)\,dx\right)^{\frac{1}{p-\varepsilon}}<\infty .$$

Here w is a weight function, i.e. an almost everywhere positive integrable function.

$L_w^{p),\theta}$ is Banach function space, nonreflexive and nonseparable. It is easy to check that the following continuous embeddings hold

$$L_w^p \hookrightarrow L_w^{p),\theta} \hookrightarrow L_w^{p-\varepsilon},\quad 0<\varepsilon<p-1 .$$

Grand Lebesgue spaces on the bounded sets of R^n were introduced by T. Iwaniec and C. Sbordone [1]. The closure of $L^p\ (1<p<\infty)$ by the norm of the grand Lebesgue spaces does not coincide with the latter space. Let us denote this closure by $\overset{\bullet}{L}{}_w^{p),\theta}$. It is known that this subspace of $L_w^{p),\theta}$ is a set of functions for which

$$\lim_{\varepsilon\to 0} \varepsilon^{\theta} \int_{\mathbb{T}} \left|f(x)\right|^{p-\varepsilon} w(x)\,dx = 0.$$

A weight function w is said to be of the Muckenhoupt class A_p $(1<p<\infty)$ if

$$\sup\left(\frac{1}{|I|}\int_I w(x)\,dx\right)\left(\frac{1}{|I|}\int_I w^{1-p'}(x)\,dx\right)^{p-1} < \infty,$$

where the supremum is taken over all intervals with length less than 2π.

For $f \in L_w^{p),\theta}$ and $w \in A_p$ we set

$$(S_h f)(x) = \frac{1}{2h}\int_{-h}^{h} f(x+t)\,dt, \quad 0<h<\pi,\ x\in\pi.$$

The operator S_h is bounded in $L_w^{p),\theta}$ uniformly by h.

The k-th order generalized moduli of smoothness is defined as

$$\Omega_k(f,\delta)_{p),\theta,w} = \sup_{\substack{0<h_i<\delta\\ 1\le i\le k}} \left\|\prod_{i=1}^{k}\left(I-S_{h_i}\right)f\right\|_{p),\theta,w}, \quad \delta>0,\ k=1,2,\ldots.$$

For $f \in L_w^{p),\theta}$ by $E_n(f)_{p),\theta,w}$ we denote the best approximation by trigonometric polynomials

$$E_n(f)_{p),\theta,w} = \inf\|f-T\|_{p),\theta,w},$$

where the infimum taken over all trigonometric polynomials T of order not greater than n.

Further by $f^{(\alpha)}$ we denote fractional derivative of order $\alpha>0$.

The following theorems are valid.

Theorem 1. *Let* $1<p<\infty$ *and* $\theta>0$. *Let* $w\in A_p$. *Then for* $f\in \overset{\bullet}{L}{}_w^{p),\theta}$ *the following inequality*

$$E_n(f)_{p),\theta,w} \le c\,\Omega_k\left(f,\frac{1}{n+1}\right)_{p),\theta,w}$$

holds with a constant $c>0$ *independent of* f *and* n.

Theorem 2. *Let* $1<p<\infty$ *and* $\theta>0$. *Let* $w\in A_p$. *If* $f^{(\alpha)}\in \overset{\bullet}{L}{}_w^{p),\theta}$ *then we have*

$$E_n(f)_{p),\theta,w} \le \frac{c}{(n+1)^{\alpha}}\,\Omega_k\left(f^{(\alpha)},\frac{1}{n+1}\right)_{p),\theta,w}$$

with a constant $c>0$ *independent of* f *and* n.

Both these theorems are Jackson type theorems.

Theorem 3. *Let* $1 < p < \infty$ *and* $\theta > 0$. *Let* $w \in A_p$. *Suppose that for* $f \in \overset{\bullet}{L}{}_w^{p),\theta}$ *we have*

$$\sum_{\nu=1}^{\infty} \nu^{\alpha-1} E_\nu (f)_{p),\theta,w} < \infty . \tag{1}$$

Then $f^{(\alpha)} \in \overset{\bullet}{L}{}_w^{p),\theta}$ *and the following estimate*

$$E_n\left(f^{(\alpha)}\right)_{p),\theta,w} \le c\left(n^\alpha E_n(f)_{p),\theta,w} + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{p),\theta,w}\right)$$

holds.

Now we focus on the inverse inequalities.

Theorem 4. *Let* $1 < p < \infty$ *and* $\theta > 0$. *Let* $w \in A_p$. *Then for* $f \in \overset{\bullet}{L}{}_w^{p),\theta}$ *we have*

$$\Omega_k\left(f^{(\alpha)}, \frac{1}{n+1}\right)_{p),\theta,w} \le c\left(\frac{1}{n^{2k}} \sum_{\nu=1}^{n} \nu^{2k+\alpha-1} E_\nu(f)_{p),\theta,w} + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{p),\theta,w}\right)$$

with a constant $c > 0$ *independent of* f *and* n.

Theorem 5. *Under the conditions of Theorem* 3 *we have the estimate*

$$\Omega_k\left(f^{(\alpha)}, \frac{1}{n+1}\right)_{p),\theta,w} \le c\left(\frac{1}{n^{2k}} \sum_{\nu=1}^{n} \nu^{2k+\alpha-1} E_\nu(f)_{p),\theta,w} + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{p),\theta,w}\right)$$

with a constant $c > 0$ *independent of* f *and* n.

Note that analogous estimates for generalized, so-called (α, ψ) derivatives due to A. I. Stepanets [4] are valid.

The direct and inverse theorems of approximations in classical function spaces were proved by S. B. Stechkin [5] and by A. F. Timan and M. F. Timan [6].

Let $f \in L^1(\mathbb{T})$ and

$$f(x) \sim \frac{a_0}{2} + \sum_{\nu=1}^{\infty} \left(a_\nu(f)\cos\nu x + b_\nu(f)\sin\nu x\right)$$

is its Fourier series.

We denote by $\sigma_n^\beta(f,x)$ $(\beta > 0)$, $\mathcal{Z}_n^k(f,x)$ and $u_r(f,x)$ the Cesáro, Zygmund and Abel-Poisson summation method, respectively. L

The following statements are true:

Theorem 6. *Let* $1 < p < \infty$ *and* $\theta > 0$. *Let* $w \in A_p$. *Then there exists a positive* c *such that for arbitrary* $f \in \overset{\bullet}{L}{}_w^{p),\theta}$ *and n the following estimate*

$$\left\| f(\cdot) - \sigma_n^\beta(f,\cdot) \right\|_{p),\theta,w} \le c\,\Omega_k\left(f, \frac{1}{n+1}\right)_{p),\theta,w}$$

holds.

Theorem 7. *Let* $1 < p < \infty$ *and* $\theta > 0$. *Let* $w \in A_p$. *Then there exists a positive* c *such that for arbitrary* $f \in \overset{\bullet}{L}{}_w^{p),\theta}$ *and* n *the inequality*

$$\left\| f(\cdot) - \sum_{\nu=1}^{n} \left(1 - \frac{\nu^k}{(n+1)^k} \right) \left(a_\nu \cos \nu x + b_\nu \sin \nu x \right) \right\|_{p),\theta,w} \le c\,\Omega_k \left(f, \frac{1}{n+1} \right)_{p),\theta,w}.$$

Theorem 8. *Let* $1 < p < \infty$ *and* $\theta > 0$. *Let* $w \in A_p$. *Then for* $f \in \overset{\bullet}{L}{}_w^{p),\theta}$ *we have*

$$\left\| f(\cdot) - u_r(f, \cdot) \right\|_{p),\theta,w} \le c\,\Omega(f, 1-r)_{p),\theta,w},$$

where c *is independent of* f *and* r, $0 < r < 1$.

In L^p $(1 < p < \infty)$ spaces, employing classical moduli of continuity, a similar problem was studied e.g. in [7] (see also the references therein).

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